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Algebra of one-particle operators for the Calogero model

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ABSTRACT

An algebra $\mathcal{G}$ of symmetric *one-particle* operators is constructed for the Calogero model. This is an infinite-dimensional Lie-algebra, which is independent of the interaction parameter λ of the model. It is constructed in terms of symmetric polynomials of raising and lowering operators which satisfy the commutation relations of the S_N-*extended* Heisenberg algebra. We interpret $\mathcal{G}$ as the algebra of observables for a system of identical particles on a line. The parameter λ, which characterizes (a class of) irreducible representations of the algebra, is interpreted as a statistics parameter for the identical particles.

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1 Introduction

This work is motivated, on one hand, by the algebraic approach to identical particles (Heisenberg quantization) [1], on the other hand, by the recent progress in understanding of the algebraic properties of the Calogero model in terms of the S_N-*extended* Heisenberg algebra [2, 3, 4]. In its general form the main idea of the algebraic approach can be formulated as searching for an algebra (the *algebra of observables* for a system of identical particles) which should have the same form independent of the statistics of particles. The statistics of particles is assumed to arise at the level of irreducible representations of the algebra of observables: different irreducible representations correspond to different statistics. Structures possessing these properties were found for two particles in one and two spatial dimensions.

For two particles on a line, the algebra of observables is the algebra $sl(2, \mathbf{R})$ [1]. The irreducible representations of the same algebra also classify the solutions of the Calogero model for two particles [5], and the algebraic approach suggests the interpretation of the singular $1/x^2$-potential of the Calogero model as a "statistical" interaction between the particles, which introduces fractional statistics in one dimension [1] (see also discussion in terms of the Schrödinger quantization in Ref. [6]). Note that the equivalence of models with $1/x^2$ interaction to systems of non-interacting particles obeying so-called exclusion statistics [7] has also been discussed recently [8].

The algebra $sl(2, \mathbf{R})$ can be constructed in terms of one-particle operators generated by the S_N-extended Heisenberg algebra of the Calogero model [4]. In this paper we use this extended Heisenberg algebra to construct an algebra of one-particle operators for an arbitrary number of particles in the Calogero model. The algebra is independent of the interaction parameter (*statistics* parameter) of the Calogero model, and the parameter then characterizes different representations of the algebra. In this way our paper can be viewed as a realization of the algebraic approach to fractional statistics for spinless identical particles on a line. Apart from this interpretation the presence of this algebra should also be of interest for the study of the algebraic structure of the Calogero model.

We should add to this that we are not able to give an expression for the parameter-independent algebra in a closed form. We rather show the existence of this algebra and give a systematic way to generate a basis for the algebra. This is supplemented by a detailed discussion of how this is done in some specific cases.

2 Definition of the algebra

We start with a system of N classical particles, with the coordinates $\{x_i\}$ $(i = 1, 2, ..., N)$ on a line. Let $\{p_i\}$ be the set of canonical momenta. Observables for a system of non-interacting identical particles should be symmetric under particle permutations and can be generated *e.g.* by the symmetric polynomials of the form $s_k = \sum_{i=1}^{N} \xi_i^k$, where $\xi_i = (x_i, p_i)$ are coordinates in the phase space [1]. Also in the quantum case the observables should be

symmetric, and they can be generated by the same symmetric polynomials, but now with x_i and p_i as operators of the N-particle system.

The operator ordering problem prevents a unique mapping between the classical and quantum cases. The algebraic relations between the observables are different in the two cases. In the classical case the symmetric polynomials define, by Poisson brackets, an infinite-dimensional Lie algebra referred to as w_∞. In the quantum case the symmetric polynomials define a commutation algebra referred to as $W_{1+\infty}$ (see *e.g.* [9]). The latter can be viewed as containing quantum corrections (higher order in $\hbar$) relative to the classical algebra w_∞. $W_{1+\infty}$ is an algebra of observables for fermions as well as bosons. However, for the case of generalized statistics which we examine here, this algebra is not represented. Instead we find a larger algebra of observables, which is homomorphic to w_∞ as well as to $W_{1+\infty}$.

To construct this algebra, we use the S_N-*extended* Heisenberg algebra which is generated by the operators a_i, $a_i^\dagger$, and K_{ij} satisfying the relations [2, 3, 4]

$$[a_i, a_j] = [a_i^\dagger, a_j^\dagger] = 0\,, \quad [a_i, a_j^\dagger] = \delta_{ij}(1 + \lambda \sum_l K_{il}) - \lambda K_{ij}\,, \tag{1}$$

$$K_{ij}K_{jl} = K_{jl}K_{il} = K_{il}K_{ij} \quad \text{for} \quad i \neq j,\; i \neq l,\; j \neq l\,, \tag{2}$$

$$K_{ij}K_{mn} = K_{mn}K_{ij} \quad \text{for} \;\; i, j, m, n \;\; \text{all different}\,, \tag{3}$$

$$K_{ij}a_j = a_i K_{ij}\,, \quad K_{ij}a_j^\dagger = a_i^\dagger K_{ij}\,. \tag{4}$$

The operators K_{ij}, which generate a representation of the symmetric group S_N, are defined for $i \neq j$, with $K_{ij} = K_{ji}$ and $(K_{ij})^2 = 1$. For convenience, we may define $K_{ii} = 0$. In terms of the above operators the Hamiltonian of the *extended* Calogero system is defined as

$$H = \frac{1}{2}\sum_{i=1}^{N}\{a_i, a_i^\dagger\}\,. \tag{5}$$

This operator has the form of a generalized system of harmonic oscillators, and the eigenvalue problem of H is easily solved, with the operators $a_i^\dagger$ as raising operators in the spectrum [4].

The original Calogero problem [10], which describes N identical particles on the line, with a harmonic oscillator potential and a ("statistical") pair interaction $V(x_i - x_j) = \lambda(\lambda-1)/(x_i - x_j)^2$, corresponds to the subspace of solutions which are totally symmetric (or totally antisymmetric) with respect to particle permutations. (To be precise, this is the form of the wave functions after a singular factor of the form $\prod_{i>j}(x_i - x_j)^\lambda$ has been factored out of the wave functions.) The symmetric functions of the operators a_i and $a_i^\dagger$ are the observables of this system.

For the discussion below it is convenient to introduce the following notation. We define a *single-particle* operator $\mathcal{A}_i$ as a linear combination of products of operators a_i and $a_i^\dagger$ with the same index, e.g. $\mathcal{A}_i = 2a_i a_i^{+2} a_i^3 + a_i^{+4} a_i^2$. *One-particle* operators are obtained from single-particle operators by summation over all particles, *i.e.* they have the form $\sum_i \mathcal{A}_i$ [1]. The one-particle operators are observables of this system of identical particles, and the algebra we seek is an algebra generated by these operators, which is independent of the statistics parameter λ. This algebra generalizes the algebra $W_{1+\infty}$ of bosons and fermions.

To construct the algebra, we first verify the following important identity

$$\sum_{ij} \mathcal{A}_j [a_i^r, a_j^\dagger] \mathcal{A}_j' = \sum_i \mathcal{A}_i r a_i^{r-1} \mathcal{A}_i' \,, \tag{6}$$

where $\mathcal{A}_i$ and $\mathcal{A}_i'$ are arbitrary single-particle operators. Starting from the last relation in (1), we derive inductively that

$$[a_i^r, a_j^\dagger] = r\delta_{ij} a_i^{r-1} + \lambda \delta_{ij} \sum_l \sum_{s=0}^{r-1} a_i^{r-1-s} a_l^s K_{il} - \lambda \sum_{s=0}^{r-1} a_i^{r-1-s} a_j^s K_{ij} \,. \tag{7}$$

Inserting (7) into the left-hand side of (6), we get

$$\sum_{ij} \mathcal{A}_j [a_i^r, a_j^\dagger] \mathcal{A}_j' = \sum_i \mathcal{A}_i r a_i^{r-1} \mathcal{A}_i' + \lambda \sum_{il} \mathcal{A}_i \sum_{s=0}^{r-1} a_i^{r-1-s} a_l^s K_{il} \mathcal{A}_i' - \lambda \sum_{ij} \mathcal{A}_j \sum_{s=0}^{r-1} a_i^{r-1-s} a_j^s K_{ij} \mathcal{A}_j' \tag{8}$$

The terms containing the operators K_{ij} are canceled, and the identity (6) follows. This identity is central for the construction of the statistics-independent algebra.

We now consider the operators of the form $L_{0n} = \sum_i a_i^n$ and $L_{m0} = \sum_i a_i^{\dagger m}$. For the commutator we find

$$[L_{0n}, L_{m0}] = \sum_{ij} [a_i^n, a_j^{\dagger m}] = \sum_{ij} \sum_{s=0}^{m-1} a_j^{\dagger s} [a_i^n, a_j^\dagger] (a_j^\dagger)^{m-s-1} = \sum_{ij} \sum_{r=0}^{n-1} a_i^r [a_i, a_j^{\dagger m}] a_i^{n-r-1} \,. \tag{9}$$

Using then the identity (6), we obtain

$$[L_{0n}, L_{m0}] = n \sum_i \sum_{s=0}^{m-1} a_i^{\dagger s} a_i^{n-1} (a_i^\dagger)^{m-s-1} = m \sum_i \sum_{r=0}^{n-1} a_i^r (a_i^\dagger)^{m-1} a_i^{n-r-1} \,. \tag{10}$$

which shows that the commutator is a one-particle operator. For the special case $n = 2$, we have

$$[L_{02}, L_{m0}] = 2m L_{m-1,1} \,, \tag{11}$$

where L_{m1} is defined by

$$L_{m1} = \frac{1}{2} \sum_i (a_i^{\dagger m} a_i + a_i a_i^{\dagger m}) \,. \tag{12}$$

It is now straightforward to show that the operators of the form L_{m0} and L_{m1} form a closed algebra. We first note that

$$[L_{m0}, L_{n0}] = 0\,, \tag{13}$$

due to the commutativity of the operators $a_i^\dagger$ with arbitrary indices. Next, using the identity (6), we get

$$[L_{m1}, L_{n0}] = nL_{m+n-1,0}\,. \tag{14}$$

Finally, for calculating the commutator $[L_{m1}, L_{n1}]$, we rearrange it, using the Jacobi identity, as follows:

$$2(m+1)[L_{m1}, L_{n1}] = [[L_{02}, L_{m+1,0}], L_{n1}] = [L_{02}, [L_{m+1,0}, L_{n1}]] - [L_{m+1,0}, [L_{02}, L_{n1}]]\,. \tag{15}$$

Then, using the identity (6) and its conjugate, and, in addition, taking into account the relation

$$\sum_i \left(a_i^{\dagger k} a_i a_i^{\dagger k'} + a_i^{\dagger k'} a_i a_i^{\dagger k} \right) = \sum_i \left(a_i a_i^{\dagger k+k'} + a_i^{\dagger k+k'} a_i \right)\,, \tag{16}$$

for k and k' non-negative integers, which is derived from (7) straightforwardly, we obtain

$$[L_{m1}, L_{n1}] = (n-m)L_{m+n-1,1}\,. \tag{17}$$

Note that Eqs. (13), (14), and (17) are the commutation relations for (the positive-frequency part of) a $U(1)$ current algebra. The Hermitian conjugation yields one more $U(1)$ current algebra with operators $L_{0m} \equiv L_{m0}^\dagger$ and $L_{1m} \equiv L_{m1}^\dagger$. The operators L_{m0} (or L_{0m}) define a $U(1)$ Kac-Moody (sub)algebra, and the operators L_{n1} (or L_{1n}) define a Virasoro (sub)algebra. For brevity we shall refer to the operators L_{m0} and L_{0m} as the Kac-Moody (KM) operators and the operators L_{m1} and L_{1m} as the Virasoro operators. Note that the Virasoro algebra generated from the S_N-extended Heisenberg algebra was also found by Bergshoeff and Vasiliev [11] and by Polychronakos [12].

The procedure of derivation of the commutation relations between the operators L_{m0} and L_{n1} is part of the general construction. We seek an algebra with generators of the form L_{mn}, where m and n, which refer to powers of the operators $a_i^\dagger$ and a_i, now can take arbitrary non-negative values. The set of operators is generated from the KM operators by repeated commutators. One building block of this construction is the observation that the commutator of a KM operator with any one-particle operator is another one-particle operator,

$$[L_{0n}, \sum_i \mathcal{A}_i] = \sum_i \mathcal{A}'_i \tag{18}$$

(with a similar expression valid for L_{m0}). This follows directly from the identity (6). Indeed, the commutator between $L_{0n} = \sum_j a_j^n$ and $\sum_i \mathcal{A}_i$ can be written as a sum of terms

where a_j^n is commuted with each factor $a_i^\dagger$ in the operator $\sum_i \mathcal{A}'_i$. For each of these terms the identity (6) can be applied. The result is a sum of one-particle operators, which is again a one-particle operator.

To proceed, we consider arbitrary 'strings' of consecutive commutators, of the form $[L_N \cdots, [L_3, [L_2, L_1]] \cdots]$, where $L_1, L_2, \ldots$ are KM operators. From the discussion above it follows that such a string is a one-particle operator. Furthermore, the set of such strings is closed under commutation, *i.e.* the commutator of two strings is a linear combination of strings. This can be shown by re-writing the commutator with the use of the Jacobi identity. The infinite-dimensional Lie algebra generated from the KM operators in this way, which will be denoted by $\mathcal{G}$, is the one we seek.

One more definition will be useful. We call an operator *mirror-symmetric* to a given product of the operators a_i and $a_i^\dagger$ if it is obtained by reversing the order of operators a_i and $a_i^\dagger$ in the product. This definition is naturally extended to one-particle operators. For example, the operator $\sum_i a_i a_i^{\dagger 3} a_i^2$ is mirror-symmetric to the operator $\sum_i a_i^2 a_i^{\dagger 3} a_i$. The commutator of a KM operator with a mirror-symmetric one-particle operator is another mirror-symmetric operator. This is readily shown by the application of identity (6). As a consequence of this, all the operators of the algebra $\mathcal{G}$ will be mirror-symmetric. One should note that only a subset of the mirror-symmetric operators will be included in the algebra. This will be shown by explicit calculation in some specific cases.

Since the algebra $\mathcal{G}$ is an algebra of one-particle operators, an arbitrary element can be written as a linear combination of operators $\sum_i A_i$, where A_i are products of the the operators a_j and $a_j^\dagger$. We say that an element of the algebra $\mathcal{G}$ is of *order* (m, n) if all A_i are products of m operators $a_i^\dagger$ and n operators a_i, and we write such an element generically as L_{mn}. If the commutators are evaluated by use of the identity (6), as outlined above, a string of KM operators will give rise to a one-particle operator of order (m, n), where

$$\begin{aligned} m &= \sum_i m_i - N + 1 \,, \\ n &= \sum_i n_i - N + 1 \,, \end{aligned} \tag{19}$$

with $(m_i, 0)$ and $(0, n_i)$ as the orders of the KM operators and N as the number of operators in the string. (A negative value of m or n then corresponds to a vanishing operator.) It follows that operators L_{mn} of all possible orders (m, n) with $m \geq 0$ and $n \geq 0$ will be generated in this way. However, several operators of the same order (m, n), but differing in the ordering of the operators $a_i^\dagger$ and a_i may be generated. To define the algebra more precisely, we have to specify the relation between these operators.

We have already noted an important identity (16) between operators with different orderings of the operators $a_i^\dagger$ and a_i. This identity reduces the number of independent operators of order $(m, 1)$ (or $(1, n)$) to one. From this identity other identities can be deduced by commutation with KM operators and by the use of (6). We note in particular

the useful identity

$$\sum_i [\mathcal{A}_i, [a_i, a_i^\dagger]] = 0\,, \tag{20}$$

This identity, which is valid for an arbitrary one-particle operator $\mathcal{A}$, can be verified either by direct calculation, or, if $\mathcal{A}$ belongs to $\mathcal{G}$, it can be derived from the identity (16) as discussed above. In fact a more general form of this identity can be given,

$$\sum_i [\mathcal{A}_i, [a_i, a_i^\dagger]^n] = 0\,, \tag{21}$$

with n as any positive integer.

When the identities are introduced, the number of linearly independent operators of given order (m, n) is reduced. Some of these identities in fact are needed to satisfy the Jacobi identity. Others may not follow from consistency of the algebra, but are imposed in order to define a *minimal* algebra consistent with the algebraic relations of the extended Heisenberg algebra. Our precise definition of $\mathcal{G}$ will be that all *statistics independent* identities between one-particle operators, which can be derived from the extended Heisenberg algebra, should be included in the definition of the $\mathcal{G}$. From studies of the low-order operators it seems that all statistics independent identities only relate operators of the same order (m, n), and in the following we shall assume that this is generally true. Other *statistics dependent* identities may possibly be derived from the extended Heisenberg algebra, but these will not be imposed on the algebra $\mathcal{G}$. Instead these identities are considered as characteristic for certain (irreducible) representations of the algebra.

We stress the point that the algebra $\mathcal{G}$, as defined above, is a statistics independent algebra. This follows from the fact that the identity (6), which is used to evaluate commutators, as well as the other identities which are used to relate operators of the same order, do not make any reference to the statistics parameter λ.

Since we cannot give a general list of generators and commutation relations of the algebra defined above, our approach will be to look for a systematic way to generate operators and commutation relations. The difficult part then will be to keep track of the identities, and thus to specify what are the independent elements of the algebra for a given order (m, n). One of the complications is that the identities relate general one-particle operators which do not necessarily belong to the algebra. This means that we have to work in a larger space of operators than is strictly needed for the definition of the algebra.

3 Spin representation

Before we turn to the question of how to construct the algebra, it is useful to introduce a classification of the operators of the algebra in terms of spin. The operators $\frac{1}{2}L_{02}$, $\frac{1}{2}L_{20}$, and$\frac{1}{2}L_{11}$ define the commutation algebra $su(1,1)$, with $\frac{1}{2}L_{02}$, $\frac{1}{2}L_{20}$ as lowering and raising operators, respectively. After a redefinition, with suitable factors of i, they can also be

viewed as defining the algebra $su(2)$. Consider the action of the operators L_{02}, L_{20}, and L_{11} on the elements g of the algebra $\mathcal{G}$ in the following way

$$L_{02}: \quad g \to [L_{02}, g] \tag{22}$$

(and similarly for L_{20} and L_{11}). If we consider the antisymmetric composition of the action of two operators of the algebra $su(2)$, *e.g.* the operators L_{02} and L_{20}, the Jacobi identity gives

$$[L_{02}, [L_{20}, g]] - [L_{20}, [L_{02}, g]] = [[L_{02}, L_{20}], g] \tag{23}$$

This shows that the commutation relations are preserved under mapping of the original operators L_{02}, L_{20}, and L_{11} into operators acting on $\mathcal{G}$. Thus, the operators define a representation of the algebra $su(1,1)$ (or $su(2)$) on the algebra $\mathcal{G}$. The algebra $\mathcal{G}$ then can be divided into subspaces corresponding to irreducible, and in fact finite dimensional, representations of given spin s.

From (7) we derive

$$\frac{1}{2}\sum_i [a_i^\dagger a_i + a_i a_i^\dagger, a_j^\dagger] = a_j^\dagger\,, \tag{24}$$

which implies the following commutator between L_{11} and any operator L_{mn} of order (m,n),

$$[L_{11}, L_{mn}] = (m-n)L_{mn}\,. \tag{25}$$

With $\frac{1}{2}L_{11}$ interpreted as the z-component of the spin operator, this gives the following spin value for an operator of order (m,n)

$$s_z = \frac{1}{2}(m-n)\,. \tag{26}$$

The operators L_{20} and L_{02} act as raising and lowering operators, respectively, and by use of these operators, one can construct multiplets of given spin. In an (m,n) diagram these multiplets are located along the diagonal with fixed $m+n$, and lie symmetrically around $m=n$. As a consequence of this, there will be one multiplet of maximal spin associated with any point (m,n) in the diagram. The maximal spin value is

$$s_{\max}(m,n) = \frac{1}{2}(m+n)\,, \tag{27}$$

and this multiplet will include the Kac-Moody as well as Virasoro operators with the given value of $m+n$.

This structure suggests the interpretation of the degeneracies at given (m,n) in terms of different spins. We thus introduce the notation $L_{nm}^{s\alpha}$ with s as an integer or half-integer smaller or equal to $s_{\max}$, given by (27). The parameter α labels different multiplets with

the same spin at a given order (m, n). (The parameter α or both parameters α and s may be dropped in cases where there is no corresponding degeneracy.)

It is worth while noting that there is no multiplet corresponding to $s = s_{\max} - 1$. This follows from the fact that the there is only one independent operator of order $(m, 1)$ (or $(1, m)$), and this operator has to belong to the multiplet $s = s_{\max}$. The maximal spin multiplet is unique, but for lower spin more than one multiplet with the same spin may appear. From studying the special representation of $\mathcal{G}$ characterized by $\lambda = 0$ we conclude that there is at least one multiplet for each of the spin values $s = s_{\max} - 2k$, with k as a positive integer smaller or equal to $s_{\max}/2$ (see Sect. 5). For $m + n \leq 5$ there is no additional degeneracy, but in the general case there will be more than one multiplet with the same spin. The degeneracies derived explicitly in Sect. 4, and listed in Fig. 1, indicate the presence of two independent spin 1 multiplets for $m + n = 6$.

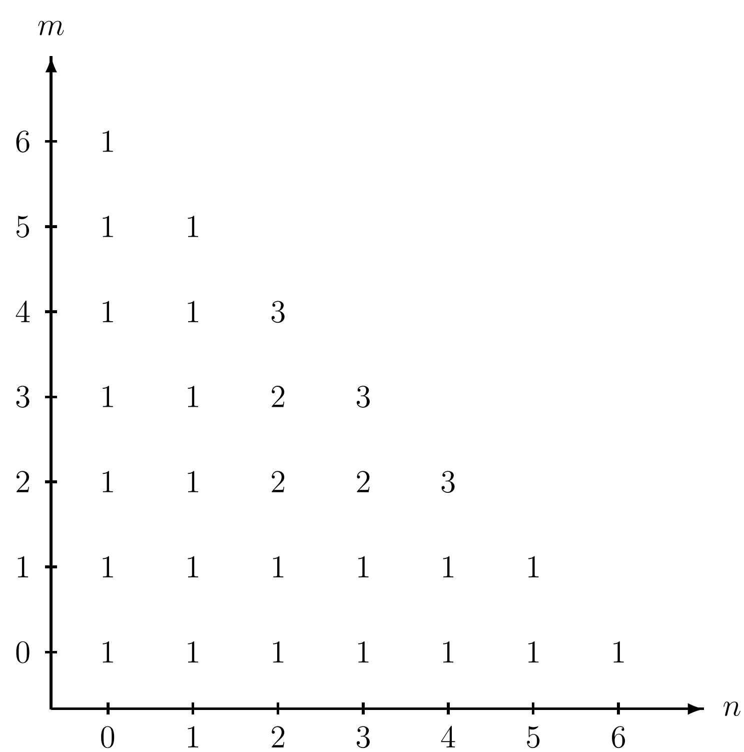

Figure 1: The degeneracies for the points (m, n) with $m + n \leq 6$.

The spin representation of the operators implies the following general form for the commutators of the algebra $\mathcal{G}$,

$$\left[L^{s\alpha}_{mn}, L^{s'\alpha'}_{m'n'}\right] = \sum_{\alpha''} \sum_{s''=|s-s'|}^{s+s'} d^{s''\alpha''}_{s\, s_{\max}\, \alpha, s'\, s'_{\max}\, \alpha'} \langle s s' s_z s'_z \mid s'' s''_z \rangle \, L^{s''\alpha''}_{m+m'-1, n+n'-1} \, . \tag{28}$$

Here $s_{\max}$ and s_z are determined by m and n through eqs. (27) and (26), and $\langle ss's_zs'_z \mid s''s''_z\rangle$ are Clebsch-Gordan coefficients. In addition to the selection rules of the Clebsch-Gordan coefficients there exist other selection rules for the coefficients $d^{s''\alpha''}_{s\, s_{\max}\, \alpha, s'\, s'_{\max}\, \alpha'}$. One of these is related to the fact that there is no multiplet corresponding to $s = s_{\max} - 1$. In Appendix B other cases of vanishing coefficients are mentioned.

In the following we will normalize the operators within one spin multiplet such that

$$[L_{02}, L^{s\alpha}_{mn}] = 2(s + s_z)L^{s\alpha}_{m-1,n+1}\,. \tag{29}$$

One should note that this gives a non-standard normalization of the Clebsch-Gordan coefficients $\langle ss's_zs'_z \mid s''s''_z\rangle$ in the commutation relations (28).

4 Constructing the algebra

In Sect. 2 the algebra $\mathcal{G}$ was defined in terms of strings of repeated commutators of KM operators. Since all the KM operators L_{m0} with $m \geq 3$ and L_{0n} with $n \geq 4$ can be generated by repeated commutators between operators from the restricted set L_{01}, L_{02}, L_{20}, and L_{03}, the algebra $\mathcal{G}$ can in fact be generated by strings of commutators involving only these four operators. It is convenient to define the subalgebra $\mathcal{G}'$ of the algebra $\mathcal{G}$ as the algebra generated by repeated commutators of only the three operators L_{02}, L_{20}, and L_{03}. One important observation is that the commutators of these three operators with the operators of order (m, n) only lead to operators of of order (m', n') with $m' + n' = m + n$ or $m' + n' = m + n + 1$. This makes it possible to generate new elements of the algebra step by step in the variable $m + n$. Thus, if the operators at level $m + n$ are known, all operators at level $m + n + 1$ can be generated by first commuting these with L_{03} and then (a finite number of times) with L_{20} and L_{02}. The remaining difficulty is to establish the identities between operators corresponding to different orderings of $a^\dagger_i$ and a_i, as discussed above. The extension of the algebra $\mathcal{G}'$ to the full algebra $\mathcal{G}$ is done simply by including the generators L_{01}, L_{10} and L_{00}. No new operators are generated to higher order (m, n), due to the simple form of the commutators between L_{01} and the operators L_{02}, L_{20}, and L_{03}.

The commutator between a KM operator and another operator of well-defined order can be viewed as a translation in the lattice formed by the points (m, n). A string of repeated operators then is represented as a path in the lattice. When constructing the operators of the algebra at a given point (m, n), it is useful to determine the independent paths, constructed from the three operators L_{02}, L_{20}, and L_{03}, which begin with either a KM or Virasoro operator and lead to the point (m, n). By *independent paths* we then mean that the operators generated by these paths cannot be related by use of the Jacobi identity and/or the commutation relations satisfied by L_{02}, L_{20} and L_{03}. This restriction to independent paths makes it possible to reduce the number of commutators to be evaluated when constructing operators at the point (m, n). For the set of operators derived in this way, the next step is to project out the spin components, and if more than one operator

of a given spin is generated at a given site, one has to check for possible linear dependence between the operators.

We have illustrated the independent paths in Fig. 2 for the special case $(m, n) = (3, 3)$. A detailed check of the identities between one-particle operators and of the linear dependences between operators belonging to the algebra $\mathcal{G}$ is performed in the Appendix A for the case $(m, n) = (2, 4)$. Note that due to identities between one-particle operators, the expressions found for the operators, in terms of a and $a^\dagger$, in general will not be unique.

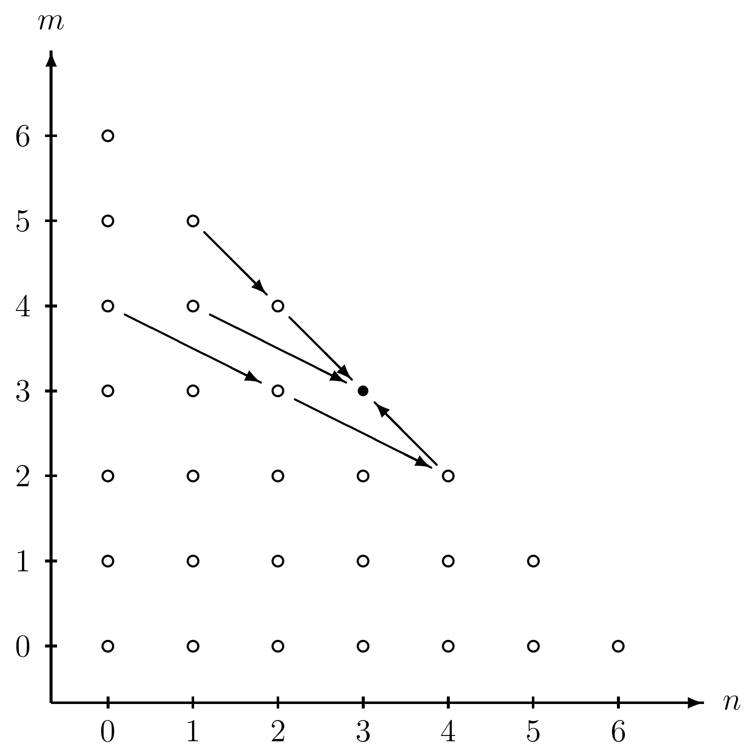

Figure 2: Three independent paths to the point $(3, 3)$.

We now examine the operators of the algebra $\mathcal{G}$ for the special cases $m + n \leq 6$. If $m = 0$ or $n = 0$ there is only one operator (the Kac Moody operator). Also at $(m, 1)$ and $(1, n)$ there is only one mirror-symmetric operator, the Virasoro operator (12). This is readily shown by use of (16). We first note that, since there is only one (trivial) operator for $m + n = 0$, there is a single spin 0 operator of this order. This operator commutes with all operators of the algebra, and we interpret it as the particle number operator. To $m + n = 1$ there corresponds a single spin 1/2 multiplet. The operators are

$$\begin{aligned} L_{10} &= a^\dagger \,, \\ L_{01} &= a \,. \end{aligned} \tag{30}$$

Here we have used a notation where the summation over particle indices is suppressed. In the following we will use this convention. This should make no confusion, since we only refer to one-particle operators. At level $m+n=2$ we have the spin 1 multiplet which we have used to define the spin content of the algebra,

$$\begin{aligned} L_{20} &= a^{\dagger 2}\,, \\ L_{11} &= \frac{1}{2}(a^{\dagger}a + aa^{\dagger})\,, \\ L_{02} &= a^{2}\,. \end{aligned} \tag{31}$$

Also the operators at $m+n=3$ are known, since only KM and Virasoro operators are included in the corresponding multiplet. This multiplet is unique and has spin 3/2,

$$\begin{aligned} L_{30} &= a^{\dagger 3}\,, \\ L_{21} &= \frac{1}{2}(a^{\dagger 2}a + aa^{\dagger 2})\,. \end{aligned} \tag{32}$$

Hereafter we only write the components of multiplets with $s_z \geq 0$. The operators with $s_z \leq 0$ are obtained (for $m+n \leq 6$) by the Hermitian conjugation:

$$L^{s}_{nm} = (L^{s}_{mn})^{\dagger}\,. \tag{33}$$

The case $m+n=4$:

In this case we need to evaluate the operators of order $(2,2)$. There are two ways to generate operators of this order, by taking the commutators $[L_{02}, L_{31}]$ and $[L_{03}, L_{30}]$. The first one generates a spin 2 operator and the second one a linear combination of a spin 2 and a spin 0 operator. We find the following expression for the spin 2 multiplet

$$\begin{aligned} L_{40} &= a^{\dagger 4}\,, \\ L_{31} &= \frac{1}{2}(a^{\dagger 3}a + ms)\,, \\ L^{2}_{22} &= \frac{1}{6}(a^{\dagger 2}a^{2} + a^{\dagger}aa^{\dagger}a + a^{\dagger}a^{2}a^{\dagger} + ms)\,. \end{aligned} \tag{34}$$

In these expressions a simplified notation is introduced, where ms denotes the mirror-symmetric operator. The spin 0 operator has the form

$$L^{0}_{22} = a^{\dagger 2}a^{2} - a^{\dagger}aa^{\dagger}a + ms\,. \tag{35}$$

The case $m+n=5$:

In this case we have to calculate the operators of orders (3,2) or (2,3). Consider order (3,2). We generate the operators of this order by taking the commutators $[L_{02}, L_{41}]$ and $[L_{03}, L_{40}]$. The first one gives a spin 5/2 operator and the second one a linear combination

of a spin 5/2 and a spin 1/2 operator. For the spin 5/2 multiplet we have the following expressions for the operators,

$$
\begin{aligned}
L_{50} &= a^{\dagger 5}\,, \\
L_{41} &= \frac{1}{2}(a^{\dagger 4}a + ms)\,, \\
L_{32}^{5/2} &= \frac{1}{4}(a^{\dagger 3}a^2 + 2a^\dagger a a^{\dagger 2}a - a^{\dagger 2}aa^\dagger a + ms)\,,
\end{aligned}
\tag{36}
$$

and after separating out the spin 1/2 components, we find for this multiplet the operators

$$
L_{32}^{1/2} = \frac{1}{2}(a^{\dagger 3}a^2 - a^{\dagger 2}aa^\dagger a + ms)\,. \tag{37}
$$

The case $m+n=6$:

We examine the independent operators of order $(3,3)$. All other operators can be derived from these by use of the lowering or raising operators. We have to consider three different commutator expressions, $[L_{02},[L_{02},L_{51}]]$, $[L_{03},L_{41}]$ and $[L_{20},[L_{03},[L_{03},L_{51}]]]$. The first one defines a spin 3 and the two others linear combinations of spin 3 and spin 1 operators. After projecting out the different spin components and checking for possible linear dependence (see Appendix A), we find that there are two independent spin 1 multiplets in addition to the spin 3 multiplet. The explicit expressions for the spin 3 operators are

$$
\begin{aligned}
L_{60} &= a^{\dagger 6}\,, \\
L_{51} &= \frac{1}{2}(a^{\dagger 5}a + ms)\,, \\
L_{42}^{3} &= \frac{1}{10}(a^{\dagger 4}a^2 + a^{\dagger 2}aa^{\dagger 2}a + 2a^\dagger aa^{\dagger 3}a + a^{\dagger 3}a^2a^\dagger + ms)\,, \\
L_{33}^{3} &= \frac{1}{20}(3a^{\dagger 3}a^3 - 3a^{\dagger 2}aa^\dagger a^2 + 2a^\dagger aa^{\dagger 2}a^2 + 2a^{\dagger 2}a^2a^\dagger a + a^\dagger aa^\dagger aa^\dagger a + 5a^\dagger a^2a^{\dagger 2}a + ms).
\end{aligned}
\tag{38}
$$

For the first spin 1 multiplet, we find

$$
\begin{aligned}
L_{42}^{1a} &= \frac{1}{6}(5a^{\dagger 4}a^2 - 7a^{\dagger 3}aa^\dagger a + 3a^{\dagger 2}aa^{\dagger 2}a + a^\dagger aa^{\dagger 3}a - 2a^{\dagger 3}a^2a^\dagger + ms)\,, \\
L_{33}^{1a} &= \frac{1}{4}(a^\dagger aa^{\dagger 2}a^2 + a^{\dagger 2}a^2a^\dagger a - 2a^\dagger aa^\dagger aa^\dagger a + ms)\,,
\end{aligned}
\tag{39}
$$

and for the second multiplet

$$
\begin{aligned}
L_{42}^{1b} &= \frac{1}{6}(3a^{\dagger 4}a^2 - 5a^{\dagger 3}aa^\dagger a + 3a^{\dagger 2}aa^{\dagger 2}a + a^\dagger aa^{\dagger 3}a - 2a^{\dagger 3}a^2a^\dagger + ms)\,, \\
L_{33}^{1b} &= -\frac{1}{6}(a^{\dagger 3}a^3 - a^{\dagger 2}aa^\dagger a^2 - a^\dagger aa^{\dagger 2}a^2 - a^{\dagger 2}a^2a^\dagger a + 2a^\dagger aa^\dagger aa^\dagger a + ms)\,.
\end{aligned}
\tag{40}
$$

The distinction between the two spin 1 multiplets is not unique. The b multiplet has been chosen to vanish for $\lambda = 0$, while the a multiplet gets a simple form in the subspace where $K_{ij} = 1$, as will be shown below.

In Appendix B we summarize some of the commutation relations which involve the low-order operators. We stress the point that the commutation relations between the operators L_{mn}, which are derived from the explicit form of these in the Calogero model, by definition characterize the (abstract) algebra $\mathcal{G}$. The operators L_{mn} of the Calogero model, in turn, can be viewed as defining a representation of this algebra. Other representations may exist which are also of interest from the point of view of generalized statistics. However, for such representations the expressions for the operators L_{mn} given above may not be valid.

The operators L_{mn} of the extended Calogero model define a *reducible* representation of $\mathcal{G}$. This follows from the fact that all operators commute with the permutations K_{ij}. Restriction of the model to a subspace which defines an irreducible representation of $\mathcal{G}$ then will correspond to a restriction of the operators K_{ij} to an irreducible representation of the permutation group. We note that such a restriction is needed when we consider the model as describing a system of *identical* particles.

When the operators L_{mn} are restricted to a subspace which defines an irreducible representation of $\mathcal{G}$, there will be more identities between the operators than those which are satisfied in the general case. We will exemplify this for the low-order operators in the irreducible representations characterized by $K_{ij} = 1$, *i.e.* for representations which are symmetric in the particle indices.

As already noticed, the spin 0 operator L_{00} can be identified with the particle number N. We find that also the second spin 0 operator in this irreducible representation is proportional to the identity operator, and can be written as a function of N and λ,

$$L^0_{22} = (1 - \lambda(\lambda - 1)(N - 1))\, N\,. \tag{41}$$

The operator L^0_{22}, in the same way as L_{00}, commutes with all elements of the algebra $\mathcal{G}$. There are relations of the same form as (41) also for the spin 1/2 and spin 1 multiplets. We have

$$L^{1/2}_{32} = (1 - \lambda(\lambda - 1)(N - 1))\, L_{21} \tag{42}$$

and

$$L^{1a}_{42} = (1 - \lambda(\lambda - 1)(N - 1))\, L_{20}\,, \tag{43}$$

with similar expressions for the other operators of the two spin multiplets. This suggests that a relation of this form may be valid also for multiplets of higher spins. For the second spin 1 multiplet we find

$$L^{1b}_{42} = -\frac{1}{3}\lambda(\lambda - 1)(N L_{20} - (L_{10})^2)\,. \tag{44}$$

We note here in particular the non-linear dependence of the spin 1/2 operator L_{10}.

5 The Bose/Fermi algebra $\mathcal{G}_0$

We now consider the special case $\lambda = 0$. This case describes bosons if the state space is restricted to states which are symmetric under interchange of particle indices and fermions if it is restricted to antisymmetric states. We make a change in notation for this special case, $a_i \to b_i$, $a_i^\dagger \to b_i^\dagger$ and $L_{mn} \to \ell_{mn}$, and the new operators b_i and $b_i^\dagger$ then satisfy the standard commutation relations

$$[b_i, b_j^\dagger] = \delta_{ij} \,. \tag{45}$$

These commutation relations imply more (linear) identities for the one-particle operators than those satisfied for general λ. In fact, a general mirror-symmetric one-particle operator of order (m, n) now can be expanded in the following way

$$\ell_{mn} = \sum_{k=0}^{k_{\max}} c_k (b^{\dagger m-2k} b^{n-2k} + ms) \,, \tag{46}$$

with

$$k_{\max}(m,n) = \left[\frac{\min(m,n)}{2} \right] \,, \tag{47}$$

where the square brackets indicate the integer part. This gives the following number of independent operators of order (m, n),

$$g(m,n) = k_{\max}(m,n) + 1 \,. \tag{48}$$

These degeneracies determine the spin multiplets which are present for a given value of $m + n$. The spin values are

$$s = s_{\max}(m,n) - 2k \,, \tag{49}$$

with $s_{\max}(m, n)$ given by (27) and $k = 0, 1, \ldots, k_{\max}(m, n)$, with only one operator for each spin. Hence for $\lambda = 0$ there is no degeneracy except the one referring to different spins, and we may drop the index α of the operators, $\ell_{mn}^{s\alpha} \to \ell_{mn}^s$.

The operators ℓ_{mn}^s define an algebra $\mathcal{G}_0$, which is smaller than the algebra $\mathcal{G}$ for general λ. We refer to this as the Bose/Fermi algebra. It is obtained from $\mathcal{G}$ by introducing the additional identities, for operators of *the same order* (m, n) and the same value of the spin s, which can be derived from the Heisenberg algebra satisfied by b_i and $b_i^\dagger$. Thus, for arbitrary λ, there are generally several multiplets of the same spin for fixed (m, n). One can choose these multiplets in such a way that all of these except one vanish for $\lambda = 0$, when the additional identities are introduced. We label the non-vanishing multiplet by letter a. The identities then can be written as

$$\ell_{mn}^{sa} = \ell_{mn}^s \,; \quad \ell_{mn}^\alpha = 0 \quad \text{for} \quad \alpha \neq a \,. \tag{50}$$

For general λ, the operators that vanish for $\lambda = 0$ form an invariant subalgebra of $\mathcal{G}$. This follows from the fact that the operators ℓ^s_{mn} satisfy the commutation relations of $\mathcal{G}$ (since this is λ-independent) as well as the commutation relations of $\mathcal{G}_0$. The algebra $\mathcal{G}_0$ can be obtained from $\mathcal{G}$ by division with this invariant subalgebra. This implies that there is a connection between the structure constants of the two algebras.

The commutation relations of the Bose/Fermi algebra $\mathcal{G}_0$ have the form,

$$\left[\ell^s_{mn}, \ell^{s'}_{m'n'}\right] = \sum_{s''=|s-s'|}^{s+s'} c^{s''}_{ss_z,s's'_z} \ell^{s''}_{m+m'-1,n+n'-1} \,. \tag{51}$$

where $c^{s''}_{ss_z,s's'_z}$ is vanish unless $s'' - s - s'$ is a positive odd integer. The structure constants can be derived from those of the algebra $W_{1+\infty}$, as we will discuss below. They in turn determine some of the structure constants (28) of $\mathcal{G}$, namely those referring only to the a-multiplets. We have

$$d^{s''a}_{s\, s_{\max}\, a, s'\, s'_{\max}\, a} \, \langle ss's_zs'_z \mid s''s''_z \rangle = c^{s''}_{ss_z,s's'_z} \,. \tag{52}$$

Thus, some of the structure constants of the full algebra $\mathcal{G}$ can be determined by calculations of the commutators in the simpler case $\lambda = 0$.

The structure constants $c^{s''}_{ss_z,s's'_z}$ do not depend on the variable $m+n$. In fact they are directly related to the structure constants of the smaller algebra $W_{1+\infty}$. To see this we note that there are more identities for $\lambda = 0$ than those valid for fixed (m,n). The expansion (46) shows that all except one operator of order (m,n) can be expressed as a linear combination of operators of lower orders. The operator which cannot be expressed in terms of operators of lower orders clearly must be the one with maximal spin, and operators with other spin values are identified with operators of the same spin but with lower values of $m+n$. We write these identifications as

$$\ell^{s,s_z} = \ell^s_{mn} \,, \qquad s_z = \frac{1}{2}(m-n) \,. \tag{53}$$

Thus, when all the linear relations between operators for $\lambda = 0$ are taken into account there is only one independent operator associated with each order (m,n), and there is only one independent spin multiplet for each integer or half integer value in the algebra. This algebra is the one which is referred to as the $W_{1+\infty}$ algebra of the Bose/Fermi system. The commutation relations of this algebra can be written as

$$\left[\ell^{s,s_z}, \ell^{s',s'_z}\right] = \sum_{s''=|s-s'|}^{s+s'} c^{s''}_{ss_z,s's'_z} \ell^{s'',s_z+s'_z} \,. \tag{54}$$

with $s' + s - s''$ odd positive, and the identification (53) then gives the corresponding expression (51) for the larger algebra $\mathcal{G}_0$. Even if the two sets of commutation relations (51) and (54) are closely related, it is of interest to note an important difference. If we

introduce Planck's constant $\hbar$, the RHS of (54) can be viewed as an expansion in powers of $\hbar$. On the other hand, the commutation relations of $\mathcal{G}_0$, in the same way as for $\mathcal{G}$, do not contain higher orders in $\hbar$.

For the operators ℓ^{s,s_z} we can find a general expression in terms of b and $b^\dagger$. To find this we make use of the commutator (29) which now can be written as,

$$[\ell^{1,-1}, \ell^{s,s_z}] = 2(s+s_z)\ell^{s,s_z-1} \,. \tag{55}$$

Starting from the operator $\ell^{s,s}$, one can use this recursively to derive the following expression for the operator ℓ^{s,s_z},

$$\ell^{s,s_z} = \sum_{m=0}^{s-s_z} \frac{(s+s_z)!(s-s_z)!}{2^m m!(s+s_z-m)!(s-s_z-m)!}(b^\dagger)^{s+s_z-m} b^{s-s_z-m} \,. \tag{56}$$

This expression can be used to derive the structure constants in (54) for the algebra $W_{1+\infty}$ in the basis of the operators ℓ^{s,s_z}. We quote here only the first term (with highest spin) in the RHS of (54),

$$[\ell^{s,s_z}, \ell^{s',s'_z}] = 2(ss'_z - s's_z)\ell^{s+s'-1,s_z+s'_z} + \cdots \tag{57}$$

The operators ℓ^{s,s_z} define a representation of the algebra $\mathcal{G}_0$. This follows from the fact that the identifications (53), $\ell^s_{m,n} = \ell^{s,s_z}$, are consistent with the commutation relations of the algebra. It is of interest to note that another set of identifications, $\ell^s_{m,n} = \ell^{s,s_z}\delta_{m,s+s_z}\delta_{n,s-s_z}$, is also consistent with the commutation relations of $\mathcal{G}_0$. These identifications correspond to keeping only the operator of the maximal spin at each point (m,n). In this case all terms in (57) except the first one vanish, and with s and s_z related to m and n as in the maximal spin multiplet, the commutation relations are reduced to that of the classical algebra w_∞,

$$[\ell_{mn}, \ell_{m'n'}] = (nm' - n'm)\ell_{m+m'-1,n+n'-1} \,. \tag{58}$$

Thus, representations of the algebras $W_{1+\infty}$ and w_∞ also provide representations of the larger algebras $\mathcal{G}_0$ and $\mathcal{G}$.

6 Conclusions

To summarize, we have shown that a parameter-independent algebra $\mathcal{G}$ of one-particle operators can be defined for the Calogero model. A basis for this algebra is generated by repeated commutators between the set of operators L_{01}, L_{02}, L_{03}, and L_{20}. We have given a recursive way to construct the algebra and used it to explicitly construct the generators for low orders. The elements of the algebra are represented in terms of spin operators, $L^{s\alpha}_{mn}$, and are grouped into spin multiplets. We have shown that in the Bose and Fermi cases one can define a smaller algebra $\mathcal{G}_0$ of a similar form as $\mathcal{G}$. This algebra is closely related to the algebra $W_{1+\infty}$, but also to the algebra w_∞ of the classical system.

We interpret the algebra $\mathcal{G}$ as the algebra of observables for a system of identical particles which satisfy generalized statistics in one dimension. The algebra is independent of particle number and of the statistics parameter. A given system of particles corresponds to an irreducible representation of the algebra. Such a representation is characterized by the particle number and by a fixed value of the statistics parameter. Identities exist between the operators in such an irreducible representation, and these identities reduce the number of independent observables. An interesting question is whether this reduction is sufficient to match the number of degrees of freedom of the (classical) system. The expressions found for low-order operators in the representations $K_{ij} = 1$ may indicate that this is in fact the case.

Since the structure of the algebra $\mathcal{G}$ is only partly known, it will be of interest to investigate this structure further. The algebra clearly can be constructed step by step in the way discussed in this paper. However the aim would be to try to find more general expressions for the algebraic relations. Let us finally mention again the question of other representations of the algebra $\mathcal{G}$, different from those found in the Calogero model. There may exist such representations which are also interesting from the point of view of generalized statistics in one dimension.

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A Appendix

Here we outline the way of finding all the linearly independent elements of the algebra $\mathcal{G}$ for the order $(2, 4)$. These operators can be generated in the following ways:

(i) starting from L_{51}, by repeated commutations with L_{02}, or equivalently, by taking the commutator of L_{20} with L_{15};

(ii) "transporting" the two linearly independent operators from the point (3,2) by taking the commutators of L_{03} with each of those operators.

This yields the three operators

$$\begin{aligned} L^{(1)} &= A + B + D + F + I\,, \\ L^{(2)} &= B + C + F\,, \\ L^{(3)} &= 2A + 2D + G + I\,, \end{aligned} \tag{59}$$

where

$$
\begin{aligned}
2A &= a^4 a^{\dagger 2} + ms\,, \\
2B &= a^3 a^\dagger a a^\dagger + ms\,, \\
2C &= a^3 a^{\dagger 2} a + ms\,, \\
2D &= a^2 a^\dagger a^2 a^\dagger + ms\,, \\
2E &= a^2 a^\dagger a a^\dagger a + ms\,, \\
2F &= a a^\dagger a^3 a^\dagger + ms\,, \\
G &= a^2 a^{\dagger 2} a^2\,, \\
H &= a a^\dagger a^2 a^\dagger a\,, \\
I &= a^\dagger a^4 a^\dagger
\end{aligned}
\tag{60}
$$

are mirror-symmetric operators of order (2,4). The set (60) includes all the mirror-symmetric operators of this order.

There are four identities between the operators (60)

$$
\begin{aligned}
B + I &= C + F\,, \\
A + F + I &= C + E + G\,, \\
A + 3E &= 3B + I\,, \\
D + F &= E + H\,,
\end{aligned}
\tag{61}
$$

which can be derived using the identity (20) as well as using the Jacobi identity to rearrange expressions with repeated commutators. Using these four identities (61), one can express the three operators (59) in terms of only five of the operators (60), *e.g.* B, C, D, E, and F. If this is a linearly independent set of operators also the operators $L^{(1)}$,$L^{(2)}$, and $L^{(3)}$ will be linearly independent. However, we will make an independent check of whether there exists any additional identity, not included among the four identities (61), which would make the three operators linearly dependent.

In the Bose/Fermi case $\lambda = 0$ one can easily show that the three operators obtained in this way are linearly dependent. This is due to an additional identity of the form

$$
B + C + 3D - 9E + 4F = 0\,. \tag{62}
$$

If the operators (59) are linearly dependent in the general case, this identity has to be valid for general λ.

To check whether the identity (62) is true or not in the general case, we consider the action of the operators (60) in the space of symmetric functions. We first extend the definition of mirror symmetry to any product of the operators a_i, $a_j^\dagger$ and K_{ij}: the mirror-symmetric expression is obtained by inversion of the order of the operators in the product. We rewrite the operators (60) in a *normal-ordered mirror-symmetric* form. For that — consider *e.g.* the operator $B = \frac{1}{2}(a^\dagger a a^\dagger a^3 + ms)$ — we rewrite the first term in

a normal-ordered form and the second, mirror-symmetric term, in an anti-normal-ordered form. Next, consider the following projection (which will be denoted by π_+): we move all the operators K_{ij} to the right of the expression and then replace K_{ij} by one. As a result, the projection of any of the operators (60) takes the form

$$\frac{1}{2}(a^{\dagger 2}a^4 + ms) + \alpha a^2 + \lambda\left[\beta a^2 + \gamma {\sum_{il}}' a_i a_l\right], \tag{63}$$

where α is independent of λ, and β and γ are linear functions of λ; the prime means summation only over non-equal i and l.

For our purposes it is sufficient to make a restriction to first order in λ in (63). Introducing the basis

$$e_0 = \frac{1}{2}(a^{\dagger 2}a^4 + ms),\;\; e_1 = \frac{1}{2}a^2,\;\; e_2 = \frac{1}{2}{\sum_{il}}'(a_i^2 + a_i a_l + a_l^2), \tag{64}$$

one obtains

$$\begin{aligned}
\pi_+(A) &= e_0\,,\\
\pi_+(B) &= e_0 - 3e_1 - \lambda e_2\,,\\
\pi_+(C) &= e_0 - 6e_1 - 2\lambda e_2 - 3\lambda(N-1)e_1\,,\\
\pi_+(D) &= e_0 - 6e_1 - 2\lambda e_2 - \lambda(N-1)e_1\,,\\
\pi_+(E) &= e_0 - 7e_1 - 3\lambda e_2\,,\\
\pi_+(F) &= e_0 - 9e_1 - 5\lambda e_2 + 3\lambda(N-1)e_1\,,\\
\pi_+(G) &= e_0 - 8e_1 - 6\lambda e_2 + 6\lambda(N-1)e_1\,,\\
\pi_+(H) &= e_0 - 8e_1 - 4\lambda e_2 + 2\lambda(N-1)e_1\,,\\
\pi_+(I) &= e_0 - 12e_1 - 6\lambda e_2\,.
\end{aligned} \tag{65}$$

With these expressions, the identities (61) projected onto the space of symmetric functions are satisfied as it should be. On the other hand, we get

$$\pi_+(B + C + 3D - 9E + 4F) = 10\lambda e_2 + 6\lambda(N-1)e_1\,, \tag{66}$$

which shows that the identity (62) holds only for $\lambda = 0$. This proves that the three operators $L^{(1)}$, $L^{(2)}$, and $L^{(3)}$ for general λ are linearly independent, *i.e.* the degeneracy at point (2,4) is $g(2,4) = 3$. Due to the obvious symmetry between the points (m,n) and (n,m), we also have $g(4,2) = 3$.

Similar considerations show that $g(3,3) = 3$. Note that three linearly independent operators of order (3,3) can be obtained by taking the commutators of L_{20} with the three linearly independent operators at point (2,4).

B Appendix

Here we discuss the commutation relations for the the spin operators introduced in Sec. 4. We start with the commutators between the operator L_{03} and operators of maximal spin. Consider first the commutator of the form $[L_{03}, L_{m0}]$ for $1 \leq m \leq 5$. It is seen from (10) that this commutator is an operator of order $(m-1, 2)$. The calculation of the spin multiplets for operators of orders (m, n) with $m + n \leq 6$ imply that operators of order $(m-1, 2)$ are linear combinations of operators of spins $\frac{1}{2}(m+1)$ and $\frac{1}{2}(m-3)$. For $1 \leq m \leq 4$, there is only one multiplet of spin $\frac{1}{2}(m-3)$. We find explicitely for this case

$$[L_{03}, L_{m0}] = 3mL_{m-1,2}^{(m+1)/2} + \frac{1}{4}m(m-1)(m-2)L_{m-1,2}^{(m-3)/2}\,. \tag{67}$$

For $m = 5$, there are two independent operators of spin $\frac{1}{2}(m-3)$ of order $(m-1, 2)$. In this case, instead of the second term in the RHS of (67), we obtain a linear combination of these two operators:

$$[L_{03}, L_{50}] = 15L_{42}^{3} + 15L_{42}^{1a} - 18L_{42}^{1b}\,. \tag{68}$$

In a similar way, we find

$$[L_{03}, L_{41}] = 12L_{33}^{3} + 6L_{33}^{1a} - \frac{36}{5}L_{33}^{1b} \tag{69}$$

and

$$[L_{03}, L_{32}^{5/2}] = 9L_{24}^{3} + \frac{3}{2}L_{24}^{1a} - \frac{9}{5}L_{24}^{1b}\,. \tag{70}$$

Next, consider commutators between the operator L_{03} and operators of a non-maximal spin. We first observe that

$$[L_{03}, L_{22}^{0}] = 0\,. \tag{71}$$

In fact, the operator L_{22}^{0} commutes with every operator of the algebra $\mathcal{G}$. This follows from the representation of the operator L_{22}^{0} in the form

$$L_{22}^{0} = N - \lambda^2 N(N-1) + \lambda \sum_{il} K_{il} \tag{72}$$

and the observation that the operator K_{il} commutes with any one-particle operator.

We also have

$$[L_{03}, L_{32}^{1/2}] = 3L_{24}^{1a} \tag{73}$$

and

$$[L_{03}, L_{23}^{1/2}] = 0\,. \tag{74}$$

The last equality reflects the selection rule implied by the spin addition formula.

Finally, we consider commutators involving the operators L_{01} and L_{10}. For all the operators L^s_{mn}, with $m+n \leq 6$, that do not vanish for $\lambda = 0$, we obtain

$$[L_{01}, L^s_{mn}] = \frac{1}{2}(2s + m - n)L^{s-1/2}_{m-1,n} \,. \tag{75}$$

On the other hand, for the operators of the multiplet (40) at level $m+n=6$, which vanish for $\lambda = 0$, we have

$$[L_{01}, L^{1b}_{mn}] = [L_{10}, L^{1b}_{mn}] = 0 \,. \tag{76}$$

Note that the last equality does not follows from the selection rule dictated by the spin addition formula. It is rather a conseqence of the fact that the operators which vanish for $\lambda = 0$ form an invarant subalgebra. The operators L^{1b}_{mn} belong to this subalgebra, and the commutator then also is an element of the subalgebra. Since the commutator is an operator of level $m+n=5$ and there are no elements of the invariant subalgebra at this level except the trivial one, the commutator has to vanish.

For low orders we also find the additional rule that the commutators between operators of spin s and s' only include operators of spins $s + s' - 1 - 2k$ with k non-negative integer (*i.e.* operators of spins differing in steps of 2). The vanishing commutator (71) is a special case of this rule. The additional selection rules that do not follow from the spin selection rules suggest the presence of more symmetry than we have established explicitly.

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